A Self-Consonance/Dissonance Approach to Anharmonicity

Luciano da Fontoura Costa
luciano@ifsc.usp.br

São Carlos Institute of Physics – DFCM/USP

June 20, 2019

Abstract

Real-world sounds always involve partials (or overtones). In this work we describe a mathematic-computational method, based on an adaptation of Helmholtz's theory of consonance, capable of providing an estimation of the self-consonance and self-dissonance of anharmonic sounds. This method considers, in addition to the given sound to be analyzed, a reference harmonic signal with the same fundamental frequency. Consonance and dissonance levels are then evaluated for every harmonic component along the reference harmonic sound, also taking into account the partial amplitudes, and respective overall indices are obtained. The methodology is illustrated with respect to nonlinearly amplified frequency-shifted sounds for increasing levels of anharmonicity, as well as regarding changes in pairwise consonance scale relationships implied by anharmonicity.

'Human music is the Concordance of divers elements in one compound, by which the spiritual nature is joined with the body...'

Micrologus, Andreas Ornithoparcus (Dowland's transl.)

1 Introduction

Sound and music are inextricably interrelated. While the latter can be understood as combinations of sounds simultaneous and sequentially, the former consists of a *seemingly* simpler concept. Indeed, at least at first sight, sounds seem to be well-defined by their *fundamental* frequency, involved *partials*, overall *volume*, and *envelope*.

Two of these properties stand out as being particularly complex: *partial content* and *envelope*. Interestingly, they can be understood as being kind of dual complements, because partials have to do with *frequency* of vibrations, while *envelope* is related to the unfolding of the amplitude of these vibrations along *time*.

In nature, it is virtually impossible to find a '*pure*' sound, in the sense of involving only its fundamental frequency, without any accompanying partial/overtone. Interestingly, also in physics and electronics it is virtually impossible to produce such pure sounds. The point is that every real-world system involves some unavoidable degree of *nonlinearity*.

Indeed, a linear system is characterized by its inability to produce new frequencies other than those already present in the input signal. Contrariwise, non-linearities intermix the frequencies in the original signal, producing an even more complex partial structure. Similarly, non-linearities in sound producing systems also contribute to making the partials much more complex not only by combining the original partials, but also by sliding frequencies up and down (e.g. with respect to the harmonic series).

All these effects ultimately converge to making real-world sounds very complex regarding their partial content and interrelationship. Yet, it is these very same partial properties that greatly contribute to defining our resulting perception of sounds.

Human perception is often relative and subjective, and may vary along time and space. Yet, there is a common set of shared features, that can be understood as defining the subsidies for human shared appreciation of sounds. In the case of two or more simultaneous sounds, their combination can be characterized in terms of the concepts of *consonance* and *dissonance* (e.g. [1, 2, 3, 4, 5, 6, 7, 8]).

As a matter of fact, almost all musical cultures and traditions, at least from ancient Indian music to modern western approaches have taken these two properties of simultaneous sounds into account. In the *Bharata* classical Indian music (e.g. [9]), for instance, alternating consonance/dissonance is an important subsidy for music composition. For instance, it is employed along a musical piece in order to enhance specific moods and effects. In tonal music, consonance provides the basis for *harmony* and *counterpoint* (e.g. [2, 10]), the two theoretical frame-

works dedicated to the study of simultaneous and sequential, respectively, combinations of notes.

It is interesting to observe that consonance and dissonance are by no means intrinsically to be sought or avoided. In fact, the appreciation of consonance and dissonance usually takes place with respect to some musical intention, such as generating surprise and diversity. For instance, the *leading-note* in tonal music (e.g. [2, 10]) is characterized by strong dissonance while being essential for priming modulations and generating 'expectation' in respective musical pieces.

Though being human-related perceptual concepts, and therefore potentially relative and subjective, consonance and dissonance can be approached in a more objective way by considering Helmholtz's respective theory (e.g. [11, 3, 12, 5, 6]).

According to this theory, consonance/dissonance stems from the combinations between the partials of the two involved sounds, especially in which regards beats. Recently [7], a simple mathematic-computational model based on Helmholtz's theory was reported capable of providing a surprising agreement with common opinion about some of the main temperaments and basic harmony. This approach considers not only matching levels between the involved partials, but also their respective intensities.

While consonance approaches can be used to infer pairwise combinations of notes, what could them contribute to characterizing, through mathematical quantifications, the tonal qualities of a *single* note or sound?

Harmonic components have been successfully applied to the characterization of anharmonicity (e.g. [13, 14, 15]). Here, we consider the possibility of extending the concept of consonance/dissonance to characterize *single notes*. This approach is aimed at providing information about how a single sound can be understood in terms of consonance and dissonance, important concepts in music and acoustics.

A key involved issue regards, given a specific sound, how to find a standard reference agains which the partials of the given sound can be compared with. A possibility is to take some specific model of partials, such as the *harmonic series* with the same fundamental as a reference.

In this work, we specifically address the latter possibility. The hypothesis here is that the harmonic series provide a kind of *prototype* distribution of partials that is characteristic of several real-world producing and reproducing sound systems.

More specifically, it could be posited that humans take the harmonic series having the same fundamental frequency as the given sound as a reference for inferring its tonal properties in terms of respective 'consonances' and 'dissonances'. The harmonic series reference tone is henceforth called *harmonic reference*. The adopted hypothesis is, therefore, directly related to the *anharmonicity* concept from classical mechanics and molecular spectroscopy (e.g [13, 16]) as well as *Inharmonicity* from music theory (e.g. [15, 17, 14]).

This article starts by revising the concept of harmonicity, and follows by discussing anharmonicity, nonlinear transformations, and frequency-shifted anharmonic signals. Then, we present the simple suggested self-consonance method based on Helmholtz's respective theory. The potential of this approach is illustrated with respect to the quantification of self-consonance/dissonance of progressively anharmonic frequency-shifted sounds as well as regarding effects on changing the consonance/dissonance patterns in respective equal temperament scales.

2 Harmonic Sounds

The *harmonic sequence* defined by the wavelengths leading to complete periods in a string is given as

$$\ell_i = \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \frac{1}{N} \tag{1}$$

The respective frequencies correspond to

$$1, 2, 3, \ldots, h_n, \ldots, N \tag{2}$$

Where 1 is the *fundamental frequency* and the remainder frequencies are commnly known as *overtones* or *partials*.

Sounds produced by *ideal* strings (linear dynamics) have respective partials obeying the harmonic sequence. if f_1 is the fundamental, the complete sound will have frequencies given as

$$h_1 f_1, h_2 f_1, h_3 f_1, \ldots, h_n f_1, \ldots, h_N f_1 \tag{3}$$

Henceforth, any sound (signal) with fundamental f_1, so that every constituent frequencies h_n (i.e. tundamental plus overtones) belong to the sequence in Equation 3, will be called an *harmonic signal*. We can classify these sounds (signals) into two main groups: (i) containing the whole set of harmonics, which we will call *complete* harmonic sounds (signals); and (ii) containing only a subset of the harmonic sequence, which will be called *incomplete* harmonic sounds (signals).

So, every complete or incomplete harmonic sound with fundamental f_1 will have each of its frequency components f_m (assumed to be ordered increasingly) obeying

$$f_m = h_n \, f_1 \tag{4}$$

for *some* $n \in \{1, 2, \ldots\}$

However, only complete harmonic sounds will satisfy

$$f_n = h_n \, f_1 \tag{5}$$

for *every* $n \in \{1, 2, \ldots\}$.

For instance, a pipe open at both ends will produce (in ideal conditions) complete harmonic sounds, while a pipe closed ar one end will generate incomplete harmonic sounds respective only to n *odd*. Sounds emanating from ideal strings are complete harmonic sounds.

3 Anharmonicity and Inharmonicity

Sounds/signals with overtone structure not fully adhering to the harmonic sequence, which is the ubiquitous case in the real-world, are henceforth called *anharmonic* in an allusion to the respective classical mechanics concept (e.g. [13]).

As hinted in the introduction section of this work, anharmonicity is related to non-linear behavior of dynamical systems. For instance, let's consider a pendulum oscillating freely from a pivot. The respective dynamics can be modeled, according to Newton's second law of movement, as

$$\ddot{\theta}(t) = -a \, sin(\theta(t)) \tag{6}$$

where $\theta(t)$ is the angular position of the bob and a is a real-valued constant dependent on the involved physical parameters (e.g. [18]). Observe that this ordinary differential equation is *non-linear* as a consequence of th $sin()$ function.

For *small angular displacements*, the above equation can be simplified as

$$\ddot{\theta}(t) = -a \, \theta(t) \tag{7}$$

which is *linear* and has as solution $\theta(t) = Asin(\omega t + \theta)$ for some real parameters A, ω, and θ (e.g [18]). Observe that this type of angular movement $\theta(t)$ involves only the fundamental frequency, without any overtone.

Let's now consider the case of *large amplitude* oscillations, which happen when we let the pendulum go from an angular position far from the vertical. In this case, the solution of the respective differential equation 6 is more ellaborate (e.g. [19]), yielding the following solution:

$$\theta(t) = 2 \, asin \left\{ s \, sn \left[K s^2 - \omega_0 t; s^2 \right] \right\} \tag{8}$$

where $sn(x; m)$, is the Jacobi's elliptic sine function, θ_0 is the initial position (therefore defining the amplitude of the oscillation) and $s = sin\,(\theta_0/2)$.

Figure 1 shows the solutions of the non-linear pendulum for several oscillation amplitudes determined by increasing respective initial positions $\theta_0 = 0.75\pi, 0.85\pi, 0.95\pi, 0.99\pi$.

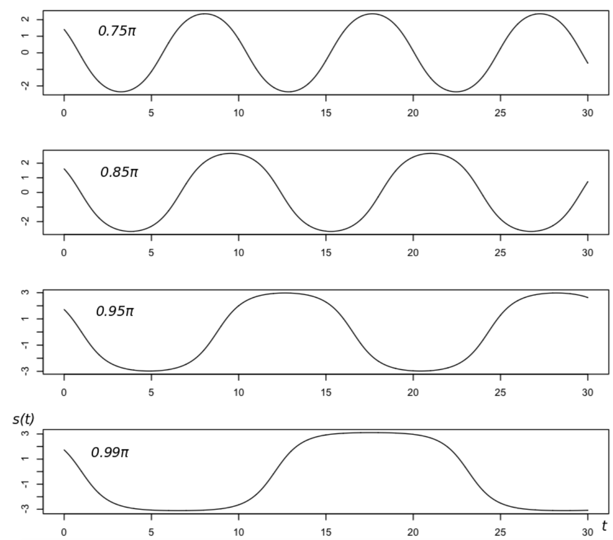

Figure 1: Angular displacements $\theta(t)$ of a nonlinear pendulum undergoing wide amplitude oscillations corresponding to initial positions $\theta_0 = 0.75\pi, 0.85\pi, 0.95\pi, 0.99\pi,$.

The resulting oscillations progressively depart from the sinusoidal function, acquiring a pronounced non-linearity as the amplitude of the oscillation is increased. These non-linearities imply several new frequency components in the resulting signal, which are not harmonic partials, hence the name *anharmonic signal*. Observe that the 'fundamental' frequency (corresponding to the largest period of the signal) also changes, decreasing with the amplitude oscillation.

The frequency shifts are often proportional to the square of the respective amplitude (e.g. [13]), i.e.

$$\Delta f \propto A^2 \tag{9}$$

In music theory, anharmonicity is alternatively called *inharmonicity* (e.g. [15, 17, 14]). It consists in an important issue, because every real-world instrument will present partials that are not perfectly harmonic. Inharmonicity studies this departure from harmonic structure unavoidably found in real-world sounds. For instance, piano inharmonicity has received great attention, as every note has 'stretched' partials (e.g. [17, 15]). The inharmonicities in the piano strings contribute to defining the characteristic 'piano sound', and therefore needs to be taken into account by piano makers and tuners.

4 Nonlinear Transformations

Frequently, the music and sounds we hear are the result of some *electronic amplification*, which is required in order

to produce electric signals with enough power to induce audible vibrations in loudspeakers or headphones.

Even if not always realized, electronic amplification intrinsically implies some unavoidable level of *nonlinear distortions*. These nonlinearities act on the partials of the input sounds, producing many new frequencies. So, a particularly interesting way for obtaining altered sounds that can be used for consonance/dissonance quantification involves producing some input sounds with harmonic partials and passing them through a non-linear amplifier.

We consider this approach in this work. For simplicity's sake, we adopt a *quadratic* nonlinearity of the type:

$$y(t) = [A\, x(t) + b]^2 \tag{10}$$

where $x(t)$ is the input signal, $y(t)$ is the respectively amplified output, and $A \in \mathrm{R}$ is the amplification factor. The constant b represents a bias often applied to input signals in some types of amplification (e.g. class A [20, 21]). This constant determines the operation point of the amplification and can be used to control the level of nonlinearity implied by the quadratic amplification. Observe that the higher the value of b, the smaller the nonlinearity will be.

Let's consider as input complete harmonic signals $x(t)$ defined by a respective fundamental frequency f_1 and N partials following the respective harmonic series, i.e. $f_1, 2f_1, 3f_1, \ldots, nf_1, \ldots, Nf_1$. The respective *power amplitude* a_i of each of these partials is assumed to follow a decaying exponential

$$a_n = e^{-\alpha n} \tag{11}$$

where $\alpha \in \mathrm{R}$. Figure 2 illustrates the power spectrum $P(f)$ of an input signal $x(t)$ considering $\alpha = 0.2$, $f_1 = 110Hz$, and $N = 20$.

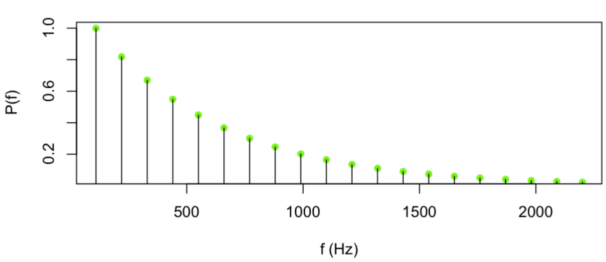

Figure 2: The power spectrum $P(f)$ of a complete harmonic sound $x(t)$ considering $\alpha = 0.2$, $f_1 = 110Hz$, and $N = 20$.

A practical way to obtain the partial content of the amplified signal is to apply the discrete Fourier transform (e.g. [22]). Figure 3 shows the square root of the power spectrum of an output signal $y(t)$ obtained from the previous input signal for $A = 2$ and $b = 1$.

Given an harmonic signal with fundamental frequency f_1, any of its polynomial transformations will necessarily

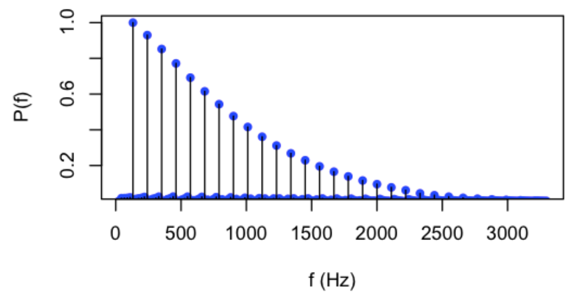

Figure 3: The square root of the power spectrum $P(f)$ of the squared previous complete harmonic sound $x(t)$, as obtained numerically.

be harmonic signals. This follows immediately from the fact that the quadratic transformation (i.e. the product of the input with itself) generates only harmonic partials corresponding to the sums and subtractions of the original harmonics, and other higher powers can be associatively understood as successive pairwise products inducing similar effects.

Interestingly, any other nonlinear transformation that can be well approximated by a power series (more formally *entire analytical functions)* will have the same property of preserving the harmonicity of the input signal. Figure 4 illustrates this fact with respect to the nonlinear transformation

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots \tag{12}$$

assuming the same input signal $x(t)$ as in the quadratic example above.

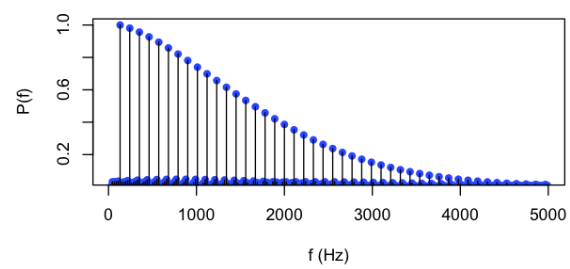

Figure 4: The square root of the power spectrum $P(f)$ of the exponential of the previous complete harmonic sound $x(t)$, as obtained numerically.

In an informal sense, the harmonic partial distribution acts as a kind of '*eigenfrequencies*' of these types of polynomial-like transformations (except for the amplitude changes and creation of new harmonic frequencies). More specifically, using the definitions described in Section 2, we have that complete/incomplete harmonic signals will be mapped into complete/incomplete harmonic signals. These types of transformations will henceforth be called *harmonic maps*.

It is important to notice that the above characterized harmonic maps is, in principle, restricted to two sounds with the *same* fundamental. In the more general case of two distinct fundamentals, new partial frequencies can be produced that can strongly influence the respective consonance/dissonance (e.g. [7]).

So, as we are dealing with given and reference sounds with the same fundamental f_1, and as harmonic nonlinear maps do not change the harmonicity of signals, we need to resort to alternative ways of producing anharmonic signals to test our consonance approach. This issue is addressed in the next section.

5 Frequency-Shifted Anharmonic Signals

Let's us now consider nonlinear transformations capable of shifting the harmonic partials. A straightforward way to do that is to act directly on the harmonic partials $h_n = 1, 2, \ldots, N$, so as to shift their position into new partials f_n. A possible manner to do that is

$$f_n = h_n^\beta \tag{13}$$

As the partials are no longer harmonic, combinations of respective frequencies as implemented by nonlinear transformations can lead to partials that are not harmonic as well as being more dispersed in the frequency domain, implying in a multiplicity of new components in the respectively obtained sounds. The higher the value of β, the larger the shifts will be.

The above approach to obtain anharmonic sounds is illustrated by the square root of the power spectrum shown in Figure 5, obtained for the quadratic amplification of the same previous harmonic signal (shown in Figure 2, with $f_1 = 110Hz$) after having its partials shifted by Equation 13 for $\beta = 1.005$.

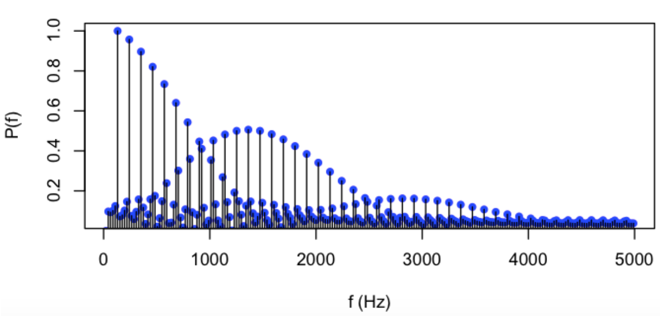

Figure 5: The square root of the power spectra of the signal shown in Figure 2, with $f_1 = 110Hz$ after having its partials shifted by Equation 13 for $\beta = 1.005$.

Now, partials appear almost everywhere in the frequency domain despite the relatively small value of β, so that the resulting signal has a substantial level of anharmonicity. In the following section we use this type of signals in order to illustrate the potential of the consonance-based approach to single sound characterization.

6 The Self-Consonance/Dissonance Method

The principle of the mathematic-computational method adopted here in order to quantify the self-consonance and self-dissonance of a given sound is outlined in Figure 6. This method consists in an adaptation of a previously reported approach aimed at estimating the consonance/dissonance between two distinct notes, which is strongly based on Helmholtz's respective theory (e.g. [11, 3, 7]).

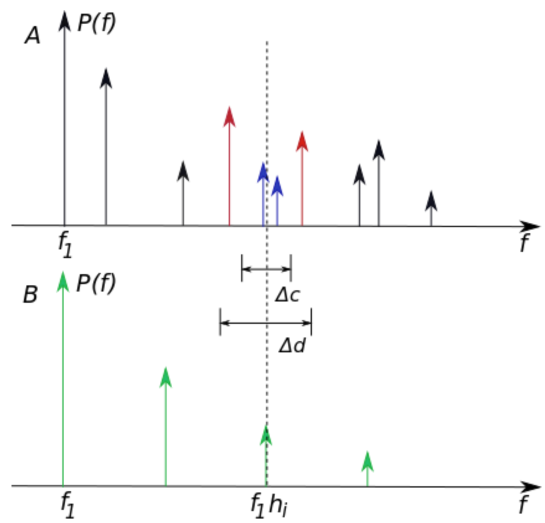

Figure 6: The framework underlying the adopted method for quantification of the 'consonance' and 'dissonance' of a given sound, whose square root spectrum is shown above (A). The harmonic reference – below, in green (B) – also with the same fundamental frequency f_1 has its harmonic components taken, one by one, as the center of two intervals with respective frequency widths Δc and Δd. Partials of the given signal B that fall within the former interval (shown in blue), i.e. Δc, are considered as 'in tune'. the partials (identified in red) that are within the interval Δd, but which are not in the previous interval, are understood as corresponding to beats. These amplitude of these partials are also taken into account in defining the respective consonance/dissonance. See text for additional information.

The sound to have its consonance/dissonance quantified (A) is henceforth identified as $s(t)$, having fundamental frequency f_1 and respective square root power spectrum $P(f)$. The first step consists in obtaining the respective harmonic counterpart signal (B), i.e. the sound that has the same fundamental f_1 but whose partials correspond to the full harmonic sequence $h_1 f_1, h_2 f_1, h_3 f_1, \ldots, h_n f_1, \ldots, h_N f_1$. A decaying profile of harmonic amplitudes is assumed, as this is often the

case with reference real-world sounds (e.g. ideal strings), accounting for the relatively less intense contribution of higher partials on the resulting consonance/dissonance.

Two frequency intervals, Δc and Δd, are henceforth employed in order to identify the partials of $s(t)$ that are here respectively understood to contribute to its self-consonance and self-dissonance. The rationale for this derives from Helmholtz's consonance theory (e.g. [11, 3, 7]) that understands the overall result of consonance and dissonance the matching (tuning) or not (beats) of partials.

Similarly to the approach reported in [7], here partial differences smaller than $2Hz$ are be understood as being relatively tuned (the partials rarely last longer than that). Diferences between $2Hz$ and $60Hz$ are here understood as potential generators of beats. So, we have that $\Delta c = 4Hz$ and $\Delta d = 30Hz$.

Now, for each of the harmonic component h_i of the reference sound, all the partials f_j of the sound A that are within the intervals Δc and Δd are identified, and the following indices are calculated so as to also take the harmonic and partial amplitudes M_i and M_j into account

$$c_i = \sum_{j \Rightarrow |f_1 h_i - f_j| \leq \Delta c} M_i M_j \, |f_1 h_i - f_j|$$

$$d_i = \sum_{\substack{j \Rightarrow |f_1 h_i - f_j| \leq \Delta d \\ |f_1 h_i - f_j| > \Delta c}} M_i M_j \, |f_1 h_i - f_j|$$

In this way, each reference harmonic h_i will be assigned respective consonance (c_i) and dissonance (d_i) indices. Alternative ways can be considered for obtaining the indices c_i and d_i, such as not considering the frequency difference as being relative to each harmonic component (e.g. $|f_1 h_i - f_j| / f_1 h_i$), considering the square root of the magnitudes, among many other possibilities. These alternatives can taken into account when adapting the consonance method to particular demands, or tuning it with specific styles, etc. Analogue comments can be made regarding the choice of the involved parameters.

Overall indices of consonance and dissonance taking into account all the N harmonics can be simply defined as

$$C = \sum_{i=1}^{N} c_i; \qquad D = \sum_{i=1}^{N} d_i$$

One of the interesting advantages of adapting Helmholtz's theory to self-consonance is that now each sound will be quantified with respect to both its consonance *and* dissonance.

7 Analysis of Anharmonic Time-Shifted Sounds

Figure 9 depicts the consonance and dissonance indices C and D obtained for the squared frequency shifted signal in the previous example considering several increasing values of β (i.e. increasing anharmonicity).

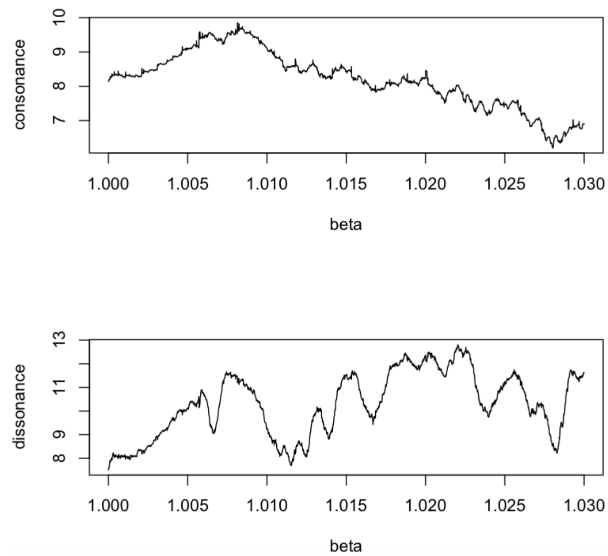

Figure 7: Overall consonance (C) and dissonance (D) obtained by the proposed method with respect to the squared frequency shifted signal in the previous example considering several increasing values of β. A coexistence of scales remindful of fractals can be observed in both signatures.

As expected, the increasing levels of anharmonicity implied by larger values of β result in decreasing consonance. An interesting signature is obtained for the dissonance, with alternating peaks and valleys. Both curves have a rather intricate structure, with coexistence of several scales that is remindful of fractal patterns. Also interesting is the fact that every sound (defined by β) involves degrees of both consonance and dissonance. A good agreement between these estimations and the auditive perception of the author has been observed.

The consonance and dissonance signatures obtained for the same case as before, but now using $b = 0$ (less linear amplification) are shown in Figure 8.

The consonance now falls much more steadily (observe that the y-axes ranges are different in Figure 9 and Figure 8), a consequence of reduced amplification linearity. The dissonance signature increases steadily to a substantially much higher level than before.

The effect of the frequency shifting and square amplification can also be inferred by visualizing the consonance levels obtained for every pair of sound relationships as defined by a given temperament. Figure 10(a) shows the consonance relationship estimated for every pair of the

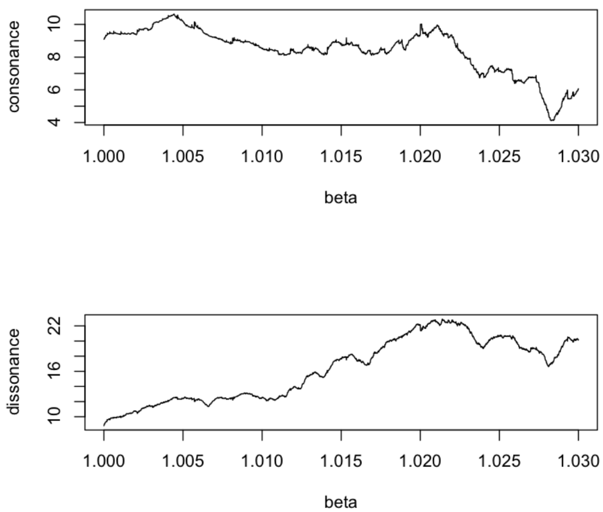

Figure 8: Overall consonance (C) and dissonance (D) for the same situation as in the previous example, but now using $b = 0$.

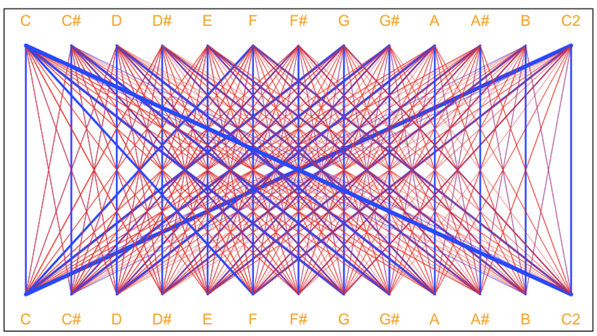

Figure 9: The pairwise consonance patterns estimate for the considered square frequency-shifted sounds as defined for the equal temperament scale. The tonic is at $110Hz$. The self-consonances are shown to 1/5 of their original values for the sake intelligibility. Consonances are shown in blue and dissonances in red, and the width of the edges reflect the intensity of the consonances.

12 notes of a equal temperament scale starting at $110Hz$. The level of consonance is reflected in the edges width. The consonance structure obtained for harmonic sounds is also shown in Figure 10(b) for reference.

The effect of the anharmonicities in changing the otherwise repetitive consonance patterns can be immediately observed. Though the expected cycle of fifths, easily discernible in (b), is much more tenuous for the anharmonic sound relationships in (a).

8 Concluding Remarks

A methodology for studying the anharmonicity of sounds in terms of self-consonance and self-dissonance based on Helmholtz's theory has been described. Given a sound to be studied, a fully harmonic signal counterpart with the same fundamental frequency is generated and its harmonic components are used for identifying respective levels of both dissonance and consonance while also taking the respective partial amplitudes into account.

While other approaches to anharmonicity can provide interesting insights to this important phenomenon, the Helmholtz theory-based method considered here allows the anharmonicity to be expressed in terms of estimations of both consonance and dissonance, which are particularly important in music theory, practice and appreciation. The application potential of the suggested methodology has been illustrated with respect to characterizing frequency-shifted signals and regarding the estimation of the effects of these alterations in changing the consonance/dissonance patterns in an equal temperament scale.

More efforts are necessary to more fully evaluate and develop the proposed simple methodology, including the consideration of other kinds of anharmonic sounds, as well as identification of parameter configurations leading to reasonable performance given specific applications.

Acknowledgments.

Luciano da F. Costa thanks CNPq (grant no. 307085/2018-0) for sponsorship. This work has benefited from FAPESP grant 15/22308-2.

References

[1] T. Christensen. *Rameau and Musical Thought in the Enlightenment.* Cambridge University Press, 1993.

[2] S. G. Laitz and C. Bartlette. *Graduate Review of Tonal Theory: A Recasting of Common-Practice Harmony, Form, and Counterpoint.* Oxford University Press, 2009.

[3] R. E. Cunningham Jr. Helmholtz's theory of consonance. `http://www.robertcunninghamsmusic.com/PDFs/Helmholtzs_Theory_of_Consonance.pdf`. Online; accessed 10-June-2019.

[4] G. Dillon. Calculating the dissonance of a chord according to Helmholtz theory. `https://arxiv.org/pdf/1306.1344.pdf`. Online; accessed 20-June-2019.

[5] R. Plomp and W. J. M. Levelt. Tonal consonance and critical bandwidth. *J. Acoust. Soc. America*, 38:548–560, 1965.

[6] W. A. Sethares. Local consonance and the relationship between timbre and scales. *J. Acoust. Soc. America*, 94:1210–1228, 1993.

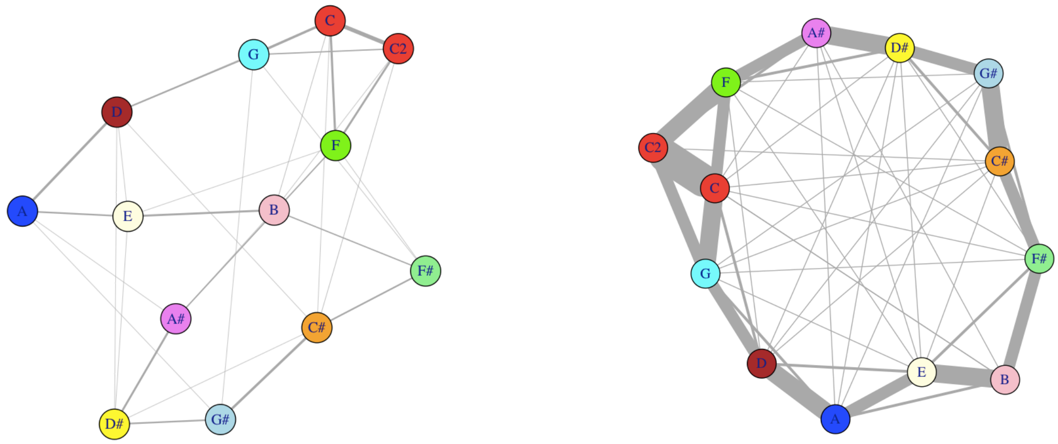

Figure 10: (a) The consonance patterns for an equal temperament scale considering the squared frequency-shifted anharmonic signals. (b) The same consonance patterns obtained for harmonic sounds. The consonance level is reflected in the edges width. Self-consonances are not shown for simplicity's sake.

[7] L. da F. Costa. Modeling consonance and its relationships with temperament and harmony. Researchgate, 2019. `https://www.researchgate.net/publication/333675642_Modeling_Consonance_and_Its_Relationships_with_Temperament_and_Harmony_CDT-10`. Online; accessed 10-June-2019.

[8] Wikipedia. Consonance and dissonance. `https://en.wikipedia.org/wiki/Consonance_and_dissonance`. Online; accessed 20-June-2019.

[9] N. A. Jairazbhoy. Harmonic implications of consonance and dissonance in ancient Indian music. *Pacific Review of Echnomusicology*, 2:28–51, 1985.

[10] S. C. Stone. *Music Theory and Composition: A Practical Approach*. Rowman and Littlefield Publ., 2018.

[11] H. von Helmholtz. *On the Sensation of Tone*. Dover, 1954.

[12] R. E. Cunningham Jr. Helmholtz' and longuet-higgins' theories of consonance and harmony. `https://www.researchgate.net/publication/2450058_Helmholtz'_and_Longuet-Higgins'_Theories_of_Consonance_and_Harmony`. Online; accessed 20-June-2019.

[13] Wikipedia. Anharmonicity. `https://en.wikipedia.org/wiki/Anharmonicity`. Online; accessed 20-June-2019.

[14] V. Valimaki J. Rauhala, H.M. Lehtonen. Fast automatic inharmonicity estimation algorithm. *J. Acoust. Soc. America*, pages EL–184, 2007.

[15] L. Daudet F. Rigaud, B. David. A parametric model and estimation techniques for the inharmonicity and tuning of the piano. *J. Acoust. Soc. America*, 133:3107–3118, 2013.

[16] N. H. Fletcher. Harmonic, anharmonic, inharmonic. *Am. J. Phys.*, pages 1205–1207, 2002.

[17] S. Hendry. Inharmonicity of piano strings. `http://www.simonhendry.co.uk/wp/wp-content/uploads/2012/08/inharmonicity.pdf`. Online; accessed 20-June-2019.

[18] L. da F. Costa. Circuits, oscillations, and the Kuramoto model as visualized by phasors. Researchgate, 2019. `https://www.researchgate.net/publication/333224636_Circuits_Oscillations_and_the_Kuramoto_Model_as_Visualized_by_Phasors_CDT-7`. Online; accessed 03-June-2019.

[19] V. Valimaki J. Rauhala, H.M. Lehtonen. Exact solution for the nonlinear pendulum. *Rev. Br. Ens. Fis.*, pages 645–648, 2007.

[20] J. M. Pettit and M. M. McWhorter. *Electronic Amplifier Circuits: Theory and Design*. McGraw-Hill, 1961.

[21] R. W. Tinnell. *Electronic Amplifiers*. Delmar Publishers, 1972.

[22] E. O. Brigham. *Fast Fourier Transform and its Applications*. Pearson, 1988.